Math 249 Lecture 7 Notes

Daniel Raban

September 8, 2017

1 Maschke's Theorem and Orthogonality of Characters

We start with a bit of review from last lecture. Given G-modules V, W, $\operatorname{Hom}(V, W)$ is a G-module under the action $g \cdot \varphi = \rho_W(g^{-1})\varphi\rho_V(g)$. Also recall that $\operatorname{Hom}(V, W)^G$ is the subspace of $\operatorname{Hom}(V, W)$ that is fixed by the action of G and that $\operatorname{Hom}_G(V, W)$ is the set of G-module homomorphisms, the maps φ such that $\rho_W(g)\varphi = \varphi\rho_V(g)$ for each $g \in G$. In this case, these are the same (since ρ_W is a homomorphism, we can take out the inverse and multiply by $\rho_W(g)$ on both sides). Last time we also showed that

$$(\chi_V, \chi_W) = \dim \operatorname{Hom}_G(V, W).$$

1.1 Maschke's theorem

Definition 1.1. The *Reynolds operator* is defined as

$$R = \frac{1}{|G|}\sum_{g \in G} g,$$

which is an element of the K-vector space KG.

The Reynolds operator R is invariant under the action of G because for $h \in G$,

$$h \cdot R = h \cdot \frac{1}{|G|}\sum_{g \in G} g = \frac{1}{|G|}\sum_{g \in G} hg = \frac{1}{|G|}\sum_{g \in G} g = R,$$

where multiplying by h just reindexes the elements of G.

We are now ready to prove Maschke's theorem.

Theorem 1.1 (Maschke)**.** *Let G be a finite group and V a G-module over a field of characteristic 0. Then V is a direct sum of irreducible G-modules.*

Proof. Let V be a G-module and $W_1 \subseteq V$ be a G-submodule. We proceed by induction on the dimension of V. It is sufficient to show that $V = W_1 \oplus W_2$ for some other G-submodule W_2, but we only have a vector subspace W_2 such that $V = W_1 \oplus W_2$. How? Let V project onto W_1 and W_2 via the maps π_1 and π_2. We have the exact sequences

$$0 \to W_2 \xrightarrow{i_2} V \to \underbrace{V/W_2}_{\cong W_1} \to 0,$$

$$0 \to W_1 \xrightarrow{i_1} V \to \underbrace{V/W_1}_{\cong W_2} \to 0,$$

where i_1, i_2 are inclusion maps. So we have $1_{W_j} = \pi_j \circ i_j$, which gives us that $W_1 \oplus \ker(\pi_1)$; this is because $v - \pi_1(v) \in \ker(\pi_1)$, so we can split each $v \in V$ into the two corresponding parts.

Note that $\pi_1 \in \mathrm{Hom}(V, W_1)$, so define $\hat{\pi}_1 = R \cdot \pi_1$. We show that $\hat{\pi}_1 \in \mathrm{Hom}(V, W_1)^G$, which makes $\hat{\pi}_1$ is a G-module homomorphism:

$$\begin{aligned} \hat{\pi}_1 \circ i_1 &= \frac{1}{|G|} \sum_g \rho_{W_1}(g) \pi_1 \rho_V(g^{-1}) i_1 \\ &= \frac{1}{|G|} \sum_g \rho_{W_1}(g) 1_{W_1} \rho_V(g^{-1}) \\ &= R \cdot 1_{W_1}, \end{aligned}$$

which is invariant under the action of G. So we have that $V \cong W_1 \oplus \ker(\hat{\pi}_1)$ as G-modules. Applying the inductive hypothesis completes the proof. $\square$

1.2 Orthogonality of characters

Lemma 1.1 (Schur)**.** *Let V and W be irreducible G-modules over an algebraically closed field K. Then*

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} K & V \cong W \\ 0 & V \not\cong W, \end{cases}$$

where the isomorphism is an isomorphism of G-modules.

Proof. Consider $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ for each $\varphi \in \mathrm{Hom}_G(V, W)$. These are G-submodules of V and W, respectively. Since V and W are irreducible, each of these is either $\{0\}$ or the whole space. If $\ker(\varphi) = V$ for all $\varphi \in \mathrm{Hom}_G(V, W)$, then $\mathrm{Hom}_G(V, W) = \{0\}$.

Otherwise, $\ker(\psi) = \{0\}$ for some $\psi \in \mathrm{Hom}_G(V, W)$, which makes $\mathrm{im} \neq \{0\}$. Then $\mathrm{im}(\psi) = W$, so ψ is a G-module isomorphism (because it is a bijection) between V and W, making $V \cong W$.

Suppose $V \cong W$. Without loss of generality, we can take $W = V$. Let $\varphi \in \mathrm{Hom}_G(V, V)$ be nonzero; then, since K is algebraically closed, φ has a nonzero eigenvector v with nonzero eigenvalue λ, so $(\varphi - \lambda I_V)v = 0$. Then $\varphi - \lambda I$ is not an isomorphism, so $\varphi - \lambda I_V = 0$ (by the irreducibility of V, which forces $\ker(\varphi) = V$). This makes $\varphi = \lambda I_V$, and since this holds for all $\varphi \in \mathrm{Hom}_G(V, W)$, $\mathrm{Hom}_G(V, W) = \mathrm{span}(\{I_V\})$. □

Theorem 1.2 (Orthogonality of characters)**.** *Irreducible characters χ_{V_i} (of irreducible G-modules V_i over an algebraically closed field K of characteristic 0) form an orthonormal basis of $\{G \to K :$ constant on conj. classes$\}$ with respect to the inner product*

$$(\chi, \varphi) = \frac{1}{|G|} \sum_g \chi(g)\varphi(g^{-1}).$$

Proof. By Schur's lemma,

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W. \end{cases}$$

This precisely means that the characters are orthonormal.

To show that the characters form a basis, note that orthogonality implies linear independence, so it suffices to show that there are at least as many irreducible characters as elements of some other basis of functions constant on conjugacy classes of G. There is a clear basis, the set of $\{\delta_{C_\alpha} : C_\alpha$ is a conjugacy class of $G\}$, where

$$\delta_{C_\alpha}(g) = \begin{cases} 1 & g \in C_\alpha \\ 0 & g \notin C_\alpha. \end{cases}$$

So it suffices to show that the number of irreducible representations is at least the number of conjugacy classes.

Consider $Z(\mathbb{C}G)$, the subspace of elements that commute with anything in $\mathbb{C}G$. If C_α is a conjugacy class of G, then $\sum_{g \in C_\alpha} g \in Z(\mathbb{C}G)$ because

$$h \left(\sum_{g \in C_\alpha} g \right) h^{-1} = \sum_{g \in C_\alpha} hgh^{-1} = \sum_{g \in C_\alpha} g,$$

where h just reindexes the elements of the sum. The elements $\sum_{g \in C_\alpha} g$ and $\sum_{g \in C_{\alpha'}} g$ are linearly independent if $C_\alpha \neq C_{\alpha'}$, and the number of these is the number of conjugacy classes of G, so the number of conjugacy classes is $\leq \dim(Z(\mathbb{C}G))$.

Take $x \in Z(\mathbb{C}G)$ with $x \neq 0$, and consider $\mathbb{C}G \circlearrowright \mathrm{Hom}(V_i, V_i)$ for some irreducible representation V_i such that $\rho_{V_i}(x)$ is not the zero map; if no such V_i exists, then $x = 0$ because $\mathbb{C}G \cong \bigoplus_i V_i^{\dim V_i} \cong \bigoplus_i \mathrm{Hom}(V_i, V_i)$. Letting $g \in G$,

$$x \cdot \rho_{V_i}(g) = \rho_{V_i}(x)^{-1} \rho_{V_i}(g) \rho_{V_i}(x) = \rho_{V_i}(x^{-1} g x) = \rho_{V_i}(g),$$

so $\rho_{V_i}(x)$ is not the zero map. Then we have a map from $Z(\mathbb{C}G) \to \bigoplus_i \mathrm{Hom}_G(V_i, V_i)$ with trivial kernel, which makes it injective. So

$$\dim(Z(\mathbb{C}G)) \leq \dim \bigoplus_i \mathrm{Hom}_G(V_i, V_i) = \sum_i \dim \mathrm{Hom}_G(V_i, V_i) = \text{number of irr. reps.},$$

since each $\mathrm{Hom}_G(V_i, V_i)$ has dimension 1. This completes the proof. $\square$

Corollary 1.1. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. The dimension of the space of functions constant on conjugacy classes of G is equal to the size of both the bases $\{\delta_{C_\alpha}\}$ and $\{\chi_{V_i}\}$. The size of the former basis is the number of conjugacy classes of G, and the size of the latter basis is the number of irreducible representations of G. $\square$

1.3 Character tables

Definition 1.2. A *character table* is a matrix (written as a table) that encodes the values of characters on conjugacy classes.

Example 1.1. The character table of S_3 is

	1	$(1\,2)$	$(1\,2\,3)$
$\chi_{\mathbb{1}} = \chi_{\ydiagram{3}}$	1	1	1
$\chi_W = \chi_{\ydiagram{2,1}}$	2	0	-1
$\chi_\varepsilon = \chi_{\ydiagram{1,1,1}}$	1	-1	1

The first two rows correspond to irreducible characters, and the third responds to the character of the defining representation of S_3 ($\sigma \mapsto P_\sigma$), which is the sum of the former two characters.